ELSEVIER Journal of Pure and Applied Algebra 128 (1998) 213–224

JOURNAL OF PURE AND APPLIED ALGEBRA

Variety of linear systems on double covering curves

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Communicated by F. Oort; received 24 November 1995; received in revised form 26 August 1996

Abstract

Irreducibility of $W_d^1(X)$ for $d \geq g - h + 1$, where X is a curve of genus g which admits a degree two map onto a general curve C of genus $h > 0$, is shown. Also the existence of a base-point-free pencil of relatively low degree on a k-gonal curves has been proved. © 1998 Elsevier Science B.V. All rights reserved.

AMS Classification: 14H45; 14H10; 14C20

0. Introduction

The aim of this paper is to study some properties of linear systems and the locus of linear systems on a complex projective algebraic curve which is a covering of another curve.

In Section 1, we prove the irreducibility of the $W_d^1(X)$ for all $d \geq g - h + 1$ on a curve X of genus g which is a double covering of a general curve C of genus $h > 0$. And this result is sharp in a sense; see Remark 1.6. In the proof of Theorem 1.1, we use the equivalence of the irreducibility of $W_d^1(X)$ and the connectivity of $W_d^1(X)$, if $W_d^1(X)$ has the positive expected dimension and is non-singular in codimension one [7]. We also use the so-called Castelnuovo–Severi inequality for a double covering X of genus g over a curve C of genus h; every base-point-free g_n^1 on X is a pull-back of a $g_{n/2}^1$ on C for any $n \leq g - 2h$ (cf. [1, Ch. 3]).

In Section 2, we consider a problem of base-point-free pencils of certain degree on a k-gonal curve as well as on a curve which is a double covering of a genus two curve.

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[1] Partially supported by MURST and GNSAGA of CNR (Italy). Major part of this work was done while he was visiting the Seoul National University under the CNR-KOSEF exchange program.

[2] Partially supported by Ministry of Education (Korea) and GARC-KOSEF.

0022-4049/98/$19.00
PII S0022-4049(97)00057-1

In proving the main results of Section 2, we use enumerative methods and computations in $H^*(C_\alpha, \mathbb{Q})$ of various sub-loci of the symmetric product C_α of the given curve C. Specifically, we compare the fundamental class of $C_\alpha^1 := \{D \in C_\alpha\colon \dim|D| \geq 1\}$ with the class of all irreducible components of C_α^1 whose general elements correspond to pencils on C with base points. This argument works because the latter components are all induced from the base curve of the covering and C_α^1 has the expected dimension. Throughout, we work over the field of complex numbers.

1. Irreducibility of $W_d^1(X)$ for double coverings

In this section we prove the following theorem.

Theorem 1.1. *Let X be a smooth algebraic curve of genus g which admits a two sheeted covering $\pi : X \to C$ onto a general curve C of genus $h > 0$, $g \geq \max\{2h^2, 5h+3\} =: \varepsilon(h)$. Then the variety $W_d^1(X)$ of pencils of degree d on X is generically reduced and irreducible with the expected dimension for all $d \geq g - h + 1$.*

Before starting to prove Theorem 1.1, we begin with the following preparatory remarks and lemmas whose proofs can be found in the related literature.

Remark 1.2 (*Coppens*; [4, Theorem 4]). Let X be an algebraic curve of genus g. Suppose that $W_d^r(X)$ has the expected dimension, i.e. $\dim W_d^r(X) = \rho(d,g,r) := g-(r+1)(g-d+r)$. Then $\dim W_{d+1}^r(X) = \rho(d+1,g,r)$ and $W_{d+1}^r(X)$ is irreducible (resp. reduced) if $W_d^r(X)$ is irreducible (resp. reduced).

The following is a well-known criteria for the irreducibility of $W_d^r(X)$ which follows from [7], Remark 1.8.

Lemma 1.3. *Let X be a smooth algebraic curve. Suppose that $W_d^r(X)$ has the expected dimension $\rho(d,g,r) > 0$ and that the codimension of the singular locus* $\operatorname{Sing} W_d^r(X)$ *is at least two. Then $W_d^r(X)$ is irreducible.*

We also need the following dimension theoretic statement for W_d^2; [5, Theorem 3.3.1].

Lemma 1.4. *Let X be a smooth algebraic curve of genus g. Let $n \in \mathbb{N}$, $g \geq 2(n+1)^2$ and* $\dim W_{n+3}^1(X) < 1$. *Then* $\dim W_d^2(X) \leq 2d-6-g$ *for $g - n < d \leq g$.*

We also have the following weaker proposition, which is an intermediate step toward the proof of Theorem 1.1 and we will prove Proposition 1.5 after finishing the proof of Theorem 1.1.

Proposition 1.5. *Let X be a smooth algebraic curve of genus g which admits a two-sheeted covering $\pi : X \to C$ onto a general curve C of genus $h > 0$, $g \geq 5h - 2$. Then the variety $W_d^1(X)$ of pencils of degree $d \geq g - h + 1$ on X is generically reduced and a general element of any component of $W_d^1(X)$ is base-point-free.*

Proof of Theorem 1.1. We first claim that $W_{g-h+1}^1(X)$ is equi-dimensional of the expected dimension $\rho(g-h+1, g, 1) = g - 2h$. Indeed, in [3, Lemma 1.2], it is proved that $W_{g-h}^1(X)$ has the expected dimension if $g \geq 4h$. Hence, the same is true for $W_{g-h+1}^1(X)$ by Remark 1.2. Therefore, by Remark 1.2 it is sufficient to prove the theorem only for $W_{g-h+1}^1(X)$.

By a result of Mayer [9], one has $\mathrm{Sing}(W_{g-h+1}^1(X)) \supset W_{g-h+1}^2(X)$. We now claim that $\dim W_{g-h+1}^2(X) \leq \rho(g-h+1, g, 1) - 2 = g - 2h - 2$: Suppose $h = 2e + 1$ is odd and take $n = h - 1 = 2e$. Then by the Castelnuovo–Severi inequality, one has

$$W_{n+3}^1(X) = W_{2e+3}^1(X) = \pi^* W_{e+1}^1(C) + W_1(X).$$

Because C is general, $\dim W_{e+1}^1(C) = \rho(e+1, h, 1) = -1$. Therefore, $\dim W_{n+3}^1(X) = \emptyset$ and hence $\dim W_{n+3}^1(X) < 1$. By taking $d = g - h + 2$ in Lemma 1.4, one has

$$\dim W_{g-h+1}^2(X) \leq \dim W_{g-h+2}^2(X) \leq 2(g - h + 2) - 6 - g = g - 2h - 2.$$

Suppose $h = 2e$ and take $n = h - 1$. Again by Castelnuovo–Severi inequality, one has

$$W_{n+3}^1(X) = W_{2e+2}^1(X) = \pi^* W_{e+1}^1(C).$$

Since C is general $\dim W_{e+1}^1(C) = \rho(e+1, h, 1) = 0$ and hence $\dim W_{n+3}^1(X) = 0 < 1$. By taking $d = g - h + 2$ in Lemma 1.4, one also has

$$\dim W_{g-h+1}^2(X) \leq \dim W_{g-h+2}^2(X) \leq 2(g - h + 2) - 6 - g = g - 2h - 2$$

and this finishes the proof of the claim.

Now, suppose that $\mathrm{Sing}\, W_{g-h+1}^1(X)$ has codimension at most one in $W_{g-h+1}^1(X)$. By the above claim we may also assume that $\mathrm{Sing}\, W_{g-h+1}^1(X) \supset W_{g-h+1}^2(X) \cup A$, where A is an irreducible closed subvariety of $W_{g-h+1}^1(X)$ such that $\dim A \geq g - 2h - 1$ and $A \not\subset W_{g-h+1}^2(X)$. We break up the proof into the following two cases.

(i) Assume that a general element of A has no base point, and choose $L \in A$ a general element. Since $L \in \mathrm{Sing}\, W_{g-h+1}^1(X)$ and by the base-point-free pencil trick, one has

$$\begin{aligned}\dim T_L W_{g-h+1}^1(X) &= \dim(\mathrm{Im}\,\mu_0)^\perp = g - 2h + \dim \mathrm{Ker}\,\mu_0 \\ &= g - 2h + h^0(X, KL^{-2}) > \dim W_{g-h+1}^1(X) = g - 2h,\end{aligned}$$

where $\mu_0 : H^0(X, L) \otimes H^0(X, KL^{-1}) \to H^0(X, K)$ is the usual cup-product map; this follows from a general theory of special linear series (cf. [2, Proposition (4.2), p. 189]). Therefore, $h^0(X, KL^{-2}) > 0$ and hence $KL^{-2} \in W_{2h-4}(X)$ for general $L \in A$. We then have

$$g - 2h - 1 \le \dim A \le \dim W_{2h-4}(X) = 2h - 4,$$

which is contradictory to the genus bound $g \ge \varepsilon(h)$.

(ii) Assume that $A \subset W^1_{g-h}(X) + W_1$, i.e. a general element of A has a base point. Note that $\dim A = g-2h-1$ since $W^1_{g-h+1}(X)$ is generically reduced by Proposition 1.5. Because $\dim W^1_{g-h}(X) = g-2h-2$, there exists a component Y of $W^1_{g-h}(X)$ such that $A = Y + W_1(X)$, $\dim Y = g - 2h - 2$.

(ii-a) Suppose Y is not of the form $Y' + W_1(X)$ for some $Y' \subset W^1_{g-h-1}(X)$. Then a general $M \in Y$ is base-point-free and $M \otimes \mathcal{O}(p)$, $p \in X$ general, has only one base point p. By the base-point-free pencil trick applied to the cup-product map

$$\mu_0 : H^0(X, M \otimes \mathcal{O}(p)) \otimes H^0(X, KM^{-1} \otimes \mathcal{O}(-p)) \to H^0(X,K),$$

$\operatorname{Ker} \mu_0 \cong H^0(X, KM^{-2} \otimes \mathcal{O}(-p)) \neq 0$ since $M \otimes \mathcal{O}(p) \in A \subset \operatorname{Sing} W^1_{g-h+1}(X)$. Therefore, we have $KM^{-2} \otimes \mathcal{O}(-p) \in W_{2h-3}(X)$ for general $M \in Y$ and $p \in X$. From this we get an inequality

$$g - 2h - 1 = \dim A \le \dim W_{2h-3}(X) + 1 = 2h - 2,$$

which is contradictory to the assumption that $g \ge \varepsilon(h)$.

(ii-b) Suppose Y is of the form $Y' + W_1(X)$ for some $Y' \subset W^1_{g-h-1}(X)$. We claim that Y is of the form $\pi^*(\Sigma^1_{n/2}(C)) + W_{g-h-n}(X)$ with $\Sigma^1_{n/2}(C)$ a component of $W^1_{n/2}(C)$, where n is even and $2[(h+3)/2] \le n \le 2h+2$.

Proof of Claim. Because Y is of the form $Y' + W_1(X)$ for some $Y' \subset W^1_{g-h-1}(X)$, Y is a component of $W^1_{g-h}(X)$ whose general element has a base point. Then $Y = \Sigma^1_n + W_{g-h-n}(X)$ for some n, $0 \le n \le g-h-1$, where Σ^1_n is a subvariety of $W^1_n(X)$ and a general element of Σ^1_n is base-point-free. We will first argue that n is relatively small compared to g. Because Y has dimension $g-2h-2$, one has $\dim \Sigma^1_n = n-h-2$, otherwise

$$\begin{aligned} g - 2h - 2 = \dim Y = \dim(\Sigma^1_n + W_{g-h-n}(X)) &\neq (n-h-2) + (g-h-n) \\ &= g - 2h - 2. \end{aligned}$$

Let L be a general element of Σ^1_n. By the standard description of the Zariski tangent space to the variety W^r_d, we have

$$\dim(\operatorname{Im} \mu_0)^\perp = \dim T_L(\Sigma^1_n) \ge \dim \Sigma^1_n \ge n - h - 2,$$

where $\mu_0 : H^0(X,L) \otimes H^0(X, KL^{-1}) \to H^0(X,K)$ is the usual cup-product map. By the base-point-free pencil trick, we have,

$$\begin{aligned} \dim(\operatorname{Im} \mu_0)^\perp &= g - \dim(\operatorname{Im} \mu_0) = g - h^0(X,L)h^1(X,L) + \dim(\operatorname{Ker} \mu_0) \\ &= g - 2(g-n+1) + h^0(X, KL^{-2}) = h^0(X, L^2) - 3 \ge n - h - 2. \end{aligned}$$

Hence, $h^0(X,L^2) \ge n-h+1$ which implies $W_{2n}^{n-h}(X) \ge n-h-2$. By reducing to pencils we have

$$\dim W^1_{n+h+1}(X) = \dim W^1_{2n-(n-h-1)}(X) \ge n-h-2+(n-h-1) = 2(n-h)-3.$$

Note that $n \le g-h-1$, thus $n+h+1 \le g$. We consider the following two cases:

(1) If $n+h+1=g$, then by passing to residual series

$$\dim W^1_{n+h+1}(X) = \dim W_{g-2}(X) = g-2 \ge 2(n-h)-3 \Leftrightarrow g \le 4h+3,$$

contradictory to the genus bound $g \ge \varepsilon(h)$.

(2) If $n+h+1 \le g-1$, we have,

$$2(n-h)-3 \le \dim W^1_{n+h+1}(X) \le (n+h+1)-2-1 \Leftrightarrow n \le 3h+1$$

by Martens' well-known theorem (cf. [2; IV, Theorem 5.1]). Thus, $n \le 3h+1 \le g-2h$ by the genus bound $g \ge \varepsilon(h)$, and again by the Castelnuovo–Severi inequality every element of Σ^1_n is a pull-back of a $g^1_{n/2}$ on C, i.e. $\Sigma^1_n = \pi^*(\Sigma^1_{n/2}(C))$, where $\Sigma^1_{n/2}(C)$ is a component of $W^1_{n/2}(C)$. Since $\dim \Sigma^1_n = \dim \pi^*(\Sigma^1_{n/2}(C)) = \dim(\Sigma^1_{n/2}(C)) = \dim(W^1_{n/2}(C)) = n-h-2 \le h$, we have $n \le 2h+2$, and $[(h+3)/2] \le n/2$ since C is general. This finishes the proof of the claim.

We next claim that $h^0(X,(\pi^*N)^{\otimes 2}) = h^0(C,N^{\otimes 2}) = n-h+1$ for $N \in \Sigma^1_{n/2}(C)$ general, $2[(h+3)/2] \le n \le 2h+2$: Since C is general, $W^1_{n/2}(C)$ is reduced at a general point $N \in \Sigma^1_{n/2}(C)$, hence $h^0(C,N^{\otimes 2}) = n-h+1$. Suppose that $h^0(X,(\pi^*N)^{\otimes 2}) > h^0(C,N^{\otimes 2})$, i.e. $\pi^* H^0(C,N^{\otimes 2}) \subsetneq H^0(X,(\pi^*N)^{\otimes 2})$. Then it follows that the complete linear system $(\pi^*N)^{\otimes 2}$ is not composed with π. Thus, X has a base-point-free g^1_x which is not composed with π, $x \le \deg(\pi^*N)^{\otimes 2} - (n-h) = 2n-(n-h) = n+h \le 3h+2$ by subtracting $n-h$ generically chosen points on X. But this is contradictory to the Castelnuovo–Severi inequality since $g \ge \varepsilon(h)$.

Now consider a general $M \in Y = \pi^*(\Sigma^1_{n/2}(C)) + W_{g-h-n}(X)$, and $M \otimes \mathcal{O}(p) \in A = Y + W_1(X)$, $p \in X$ general. Then $M = \pi^*N \otimes \mathcal{O}(p_1 + \cdots + p_{g-h-n})$ and $M \otimes \mathcal{O}(p) = \pi^*N \otimes \mathcal{O}(p_1 + \cdots + p_{g-h-n} + p)$, where $N \in \Sigma^1_{n/2}(C)$. Applying the base-point-free pencil trick to the cup-product map $\mu_0 : H^0(X, M \otimes \mathcal{O}(p)) \otimes H^0(X, KM^{-1} \otimes \mathcal{O}(-p)) \to H^0(X,K)$, one has $\operatorname{Ker}\mu_0 \cong H^0(X, K \otimes (\pi^*N)^{\otimes -2} \otimes \mathcal{O}(-p_1 - \cdots p_{g-h-n} - p))$. On the other hand, from the previous claim $h^0(X,(\pi^*N)^{\otimes 2}) = n-h+1$ and hence $h^0(X, K \otimes (\pi^*N)^{\otimes -2}) = g-n-h$. Since $p_1, \ldots, p_{g-h-n}, p \in X$ have been chosen generically, we have $\dim \operatorname{Ker}\mu_0 = h^0(X, K \otimes (\pi^*N)^{\otimes -2} \otimes \mathcal{O}(-p_1 - \cdots p_{g-h-n} - p)) = 0$. But this is contradictory to the fact that $M \otimes \mathcal{O}(p) \in \operatorname{Sing} W^1_{g-h+1}(X)$, i.e. $\dim \operatorname{Ker}\mu_0 > 0$.

So far, we have shown that $\operatorname{Sing} W^1_{g-h+1}(X)$ has codimension at least two in $W^1_{g-h+1}(X)$. By [7, Remark 1.8] we finally conclude that $W^1_{g-h+1}(X)$ is irreducible.

We now finish the first section with the proof of Proposition 1.5.

Proof of Proposition 1.5. By Remark 1.2, it is enough to prove Proposition 1.5 for $d = g-h+1$. Let Σ be a component of $W^1_{g-h+1}(X)$ whose general element has a base point. Set $\Sigma = \Sigma_n^1 + W_{g-h+1-n}(X)$ for some $n \le g - h$, where Σ_n^1 is a subvariety of $W_n^1(X)$ whose general element is base-point-free. Hence, $\dim \Sigma_n^1 = \dim \Sigma - (g - h + 1 - n) = n - h - 1$. Let $L \in \Sigma_n^1$ be general. Then $\dim T_L \Sigma_n^1 \ge \dim \Sigma_n^1 \ge n - h - 1$, and hence by the base-point-free pencil trick,

$$\begin{aligned}\dim(\operatorname{Im}\mu_0)^{\perp} &= g - h^0(X,L)h^1(X,L) + \dim \operatorname{Ker}\mu_0\\ &= g - 2(g-n+1) + h^0(X,KL^{-2}) = h^0(X,L^2) - 3 \ge n - h - 1.\end{aligned}$$

Then $h^0(X,L^2) = n - h + 2$ for general $L \in \Sigma_n^1$ and, hence, $\dim W_{2n}^{n-h+1} \ge n - h - 1$. By taking off $(n - h)$ general points on X, we have,

$$\dim W_{n+h}^1(X) \ge 2(n-h) - 1. \tag{1.5.1}$$

Note that $n + h \le g$ and we distinguish the following two cases.

(i) If $n + h = g$, $\dim W_g^1(X) = \dim W_{g-2}(X) = g - 2 \ge 2(n-h) - 1 = 2(g - 2h) - 1$, which is contradictory to the genus bound $g \ge 5h - 2$.

(ii) If $n + h \le g - 1$, $2(n-h) - 1 \le \dim W_{n+h}^1(X) \le n + h - 3$ by (1.5.1) and by Martens theorem. Then by the genus bound $g \ge 5h - 2$, $n \le 3h - 2 \le g - 2h$ and hence by Castelnuovo–Severi inequality, one has $\Sigma_n^1 \subset \pi^*(W_{n/2}^1(C))$. On the other hand, $\dim W_{n/2}^1(C) = n - h - 2$ since, C is general. Hence, $n - h - 1 = \dim \Sigma_n^1 \le \dim \pi^*(W_{n/2}^1(C)) = n - h - 2$ which is a contradiction. And this proves that a general element of any component of $W_d^1(X)$ is base-point-free.

For the generically reducedness of $W_{g-h+1}^1(X)$, we only need to compute the dimension of the Zariski tangent space $T_L W_{g-h+1}^1(X)$ at a general L. Suppose $\dim T_L W_{g-h+1}^1(X) = \dim(\operatorname{Im}\mu_0)^{\perp} > \dim W_{g-h+1}^1(X) = g - 2h$ for a general $L \in W_{g-h+1}^1(X)$. L being base-point-free, it follows that $h^0(X, KL^{-2}) \ge 1$ for general $L \in W_{g-h+1}^1(X)$ by the base-point-free pencil trick. Then we have $g - 2h \le \dim W_{g-h+1}^1(X) \le \dim W_{2h-4} = 2h - 4$, which is contradictory to the genus bound $g \ge 5h - 2$. □

Remark 1.6. (i) Note that the result of Theorem 1.1 is sharp. Indeed, in the course of the proof of Theorem 0.1 in [3], it has been shown that $W_{g-h}^1(X)$ is reducible for the double covering X of a general curve C.

(ii) We proved Theorem 1.1 under the assumption that C is a general curve of genus h. In fact, we only need the condition that the schemes $W_k^1(C)$ satisfy the expected dimension and emptiness from Brill–Noether theory.

2. Existence of base-point-free pencils on k-gonal curves

In this section, we consider a problem concerning the existence of complete base-point-free pencils of certain degree on a k-gonal curve as well as on a curve which admits a double covering onto a curve C of genus 2. We first remark the following general fact which has been known already (cf. [6, Theorem (2.2.2), Corollary (2.2.3) and Theorem (3.1)]).

Remark 2.1. On a general k-gonal curve C of genus g, $3 \le k < [(g+3)/2]$, there exists a complete base-point-free pencil g^1_n on C such that $2g^1_n$ is non-special for any $n \in \mathbb{N}$ with $g/2+1 \le n \le g$. Furthermore, if $k>3$ and $g/2+1 \le n \le g-1$ then there exists a primitive complete g^1_n on a general k-gonal curve.

For an arbitrary given k-gonal curve admitting a simple g^1_k, we have the following preliminary theorem.

Theorem 2.2. *Let C be a k-gonal curve of genus $g>(3k-6)(k-1)$ with a simple g^1_k. Then there exists a complete base-point-free pencil of degree n for any $n \ge g+2-k$.*

For the proof of the above theorem we need to invoke the following theorem of Coppens concerning the variety of special linear systems on algebraic curves [4].

Theorem 2.3 (Coppens [4–6]). *Let C be an algebraic curve of genus g. If* $\dim W^1_d(C) = d-2-j$ *for some $j+3 \le d \le g-1-j$ $(j \ge 0)$ and $g \ge (2j+1)(j+1)$ then* $\dim W^1_{j+3}(C) = 1$.

Lemma 2.4. *Let C be a k-gonal curve of genus $g, g \ge (2k-5)(k-2)$. Suppose that* $\dim W^1_k(C) = 0$. *Then $W^1_{g+2-k}(C)$ has the expected dimension $g-2k+2 = \rho(g+2-k, g, 1)$.*

Proof. Suppose $\dim W^1_{g+2-k}(C) \ge g-2k+3$ and set $\dim W^1_{g+2-k}(C) = (g+2-k)-2-j$. Then $j \le k-3$ and the numerical hypothesis in Theorem 2.3 is satisfied for $d = g+2-k$. Thus, if $g \ge (2k-5)(k-2)$, $\dim W^1_{j+3}(C) = 1$ contrary to the hypothesis $W^1_k(C) = 0$. $\square$

Proof of Theorem 2.2. Since the existing g^1_k is simple and by the assumption on the genus g, g^1_k is unique. Clearly $W^1_k(C) + W_{g+2-2k}(C)$ is an irreducible component of $W^1_{g+2-k}(C)$, by Lemma 2.4. Let ω^1_{g+2-k} be the fundamental class of $W^1_{g+2-k}(C)$ in $J(C)$, the Jacobian variety of C and let ω be the class of $W^1_k(C) + W_{g+2-2k}(C)$. Because $W^1_{g+2-k}(C)$ is of pure dimension $\rho(g+2-k, g, 1)$ by Lemma 2.4, one can compute the class ω^1_{g+2-k}; Theorem (1.3) in [2, p. 212]. Also the class ω can be computed by Poincaré's formula; [2, p. 25]. Hence we have

$$\omega^1_{g+2-k} = \frac{1}{k!(k-1)!}\theta^{2k-2} \quad \text{and} \quad \omega = \frac{1}{(2k-2)!}\theta^{2k-2},$$

where θ denotes the class of the theta divisor in $J(C)$. Thus,

$$\omega^1_{g+2-k}\theta^{g-2k+2} = \frac{1}{k!(k-1)!}\theta^{2k-2}\theta^{g-2k+2} = \frac{1}{k!(k-1)!}\theta^{g} = \frac{g!}{k!(k-1)!}$$
$$\ne \omega\theta^{g-2k+2} = \frac{\theta^g}{(2k-2)!} = \frac{g!}{(2k-2)!}.$$

On the other hand, we remark that $W^1_{g+2-k}(C)$ is reduced at a general point $A := g^1_k \otimes \mathcal{O}(\Delta)$ of $W^1_k(C) + W_{g+2-2k}(C)$, $\mathcal{O}(\Delta) \in W_{g+2-2k}(C)$; this follows from the description of the tangent space to $W^1_{g+2-k}(C)$ at A (cf. [2, Prop. (4.2), p. 189]) and the fact that $h^0(C, KA^{-2}(\Delta)) = 0$, which can be computed easily, or from a remark at p. 189, after Theorem 4, in [4]. Therefore, we deduce that there exists a component in $W^1_{g+2-k}(C)$ other than the component $W^1_k(C) + W_{g-2k+2}(C)$, hence $W^1_{g+2-k}(C)$ is reducible. We will now show that $W^1_k(C) + W_{g-2k+2}(C)$ is the only component of $W^1_{g+2-k}(C)$ whose general element has a base point. Let Γ be a component of $W^1_{g+2-k}(C)$ whose general element has a base point. Then $\Gamma = \Gamma^1_e + W_{g+2-k-e}(C)$, $k \le e \le g+1-k$, where Γ^1_e is a component of $W^1_e(C)$ whose general element is base-point-free. Assume $e \ne k$. We first note that $\dim \Gamma^1_e \ge e-k$ otherwise $2(g+1-k) - g = \rho(g+2-k, g, 1) \le \dim \Gamma = \dim \Gamma^1_e + \dim W_{g+2-k-e}(C) \le e-k-1+(g+2-k-e) = g+1-2k$, which is absurd. Let L be a general element of Γ^1_e. Again by the description of the Zariski tangent space to Γ^1_e at L, $\dim(\operatorname{Im}\mu_0)^\perp = \dim T_L\Gamma^1_e \ge \dim \Gamma^1_e \ge e-k$, where μ_0 is the usual cup-product map with respect to L. On the other hand, by the base-point-free pencil trick we have

$$\dim(\operatorname{Im}\mu_0)^\perp = g - \dim(\operatorname{Im}\mu_0) = g - 2(g-e+1) + \dim(\operatorname{Ker}\mu_0)$$
$$= 2e - 2 - g + h^0(C, KL^{-2}).$$

Hence, $h^0(C, L^2) \ge e-k+3$, which implies $W^{e-k+2}_{2e}(C) \ge e-k$.

By recalling the fact that $\dim W^{r-1}_{d-1}(C) \ge \dim W^r_d(C) + 1$ we have

$$\dim W^1_{e+k-1}(C) \ge \dim W^{e-k+2}_{2e}(C) + e - k + 1$$
$$\ge (e-k) + (e-k+1) = 2(e-k) + 1.$$

We want to apply the Martens' dimension theorem to this situation: We first note that $e \le g+1-k$ and hence $e+k-1 \le g$.

(i) In case $e+k-1 = g$, by passing to the residual series, $\dim W^1_{e+k-1}(C) = \dim W_{g-2}(C) = g-2 \ge 2(e-k)+1$ and hence $g \le 4k-5$, contrary to the assumption on the genus g.

(ii) In case $e+k-1 \le g-1$, we have $2e-2k+1 \le \dim W^1_{e+k-1}(C) \le (e+k-1)-2-1$ and hence $e \le 3k-5$. Since g^1_k is simple and $g > (k-1)(3k-6)$ this is a contradiction.

Thus, the only possibility is $e = k$. Since $W^1_{g+2-k}(C)$ is reducible, there exists complete base-point-free pencils of degree $g+2-k$ on C corresponding to elements of components other than $W^1_k(C) + W_{g-2k+2}(C)$.

For any n with $g+2-k \le n \le g$, by the excess linear series argument [8], it follows that $\dim W^1_n(C) = \rho(n, g, 1) = 2(n-1) - g$. Thus there exist complete base-point-free pencils $g^1_g, g^1_{g-1}, \ldots, g^1_{g+2-k}$. □

In the next theorem, we proceed one step further to obtain the following result of the existence of complete base-point-free pencil of degree $g+1-k$ on a k-gonal curve with a simple g^1_k.

Theorem 2.5. *Let C be a k-gonal curve with a simple g^1_k of genus $g > \pi(k^2, 2k-2)$ where $\pi(d,r) := m(d-1-\frac{1}{2}(m+1)(r-1))$, $m = [(d-1)/(r-1)]$. Then there exists a complete base-point-free pencil of degree $g+1-k$.*

Proof. First note that $W^1_{g+1-k}(C)$ cannot have the expected dimension; otherwise $g+1-2k \le \dim W^1_k(C) + \dim W_{g+1-2k}(C) \le \rho(g+1-k, g, 1) = 2(g-k)-g$. One can then apply Theorem 2.3 to show that $\dim W^1_{g+1-k}(C) = g+1-2k$, which implies $W^1_k(C) + W_{g+1-2k}(C)$ is indeed a component of $W^1_{g+1-k}(C)$. Also one can follow the argument as in the proof of (2.1.1) of [6] to show that $W^1_k(C) + W_{g+1-2k}(C)$ is the only component of dimension $g+1-2k$.

On the other hand, one can show that $W^1_k(C) + W_{g+1-2k}(C)$ is the only component of $W^1_{g+1-k}(C)$ whose general element has a base point by using the same argument as in the proof of the assertion that $e = k$ in the previous theorem. Thus it remains to prove that $W^1_{g+1-k}(C)$ is reducible, which will complete the proof of the theorem.

Assume that $W^1_{g+1-k}(C)$ is irreducible, i.e. $W^1_{g+1-k}(C) = W^1_k(C) + W_{g+1-2k}(C)$. For any $g-k^2+2k-3$ points on C, say, $P_1 \cdots P_{g-k^2+2k-3}$, one apparently has

$$\begin{aligned}|(k-1)g^1_k + P_1 + \cdots + P_{g-k^2+2k-3}| &\in W^{k-1}_{g+k-3}(C)\\ &= \kappa - W^1_{g+1-k}(C) = \kappa - (W^1_k(C) + W_{g+1-2k}(C)),\end{aligned}$$

where κ denotes the point on the Jacobian $J(C)$ corresponding to the canonical divisor K on C. Thus there exists $Q_1, \ldots, Q_{g+1-2k}$ on C such that

$$\begin{aligned}\dim|kD + P_1 + \cdots + P_{g-k^2+2k-3}| &= \dim|K - Q_1 - \cdots - Q_{g+1-2k}|\\ &= 2k-2+\dim|Q_1 + \cdots + Q_{g+1-2k}| \ge 2k-2, \qquad (2.5.1)\end{aligned}$$

where $D \in g^1_k$. On the other hand, if $h^0(C, |K-kD|) = h^0(C, |kD|) + g - k^2 - 1 < g - k^2 + 2k - 3$ then there exists $R_1, \ldots, R_{g-k^2+2k-3} \in C$ such that $\dim|K - kD - R_1 - \cdots - R_{g-k^2+2k-3}| = -1$ and hence $\dim|kD + R_1 + \cdots + R_{g-k^2+2k-3}| = 2k-3$. But this is contradictory to the inequality (2.5.1). Therefore we have $\dim|kD| \ge 2k-2$. Let f be the morphism of degree m onto a curve C' of degree k^2/m in $\mathbb{P}^\alpha$ associated with $|kD|$ where $\alpha = \dim|kD| \ge 2k-2$. By the Riemann–Roch theorem applied to the induced series of degree k^2/m and of dimension α on C', we have $2k-2 \le \alpha \le k^2/m$, whence $m < k$. Since $\dim|kg^1_k - g^1_k| = \dim|(k-1)g^1_k| \ge 0$ the map $C \to \mathbb{P}^1$ given by the g^1_k factors through f. By the assumption that g^1_k is simple we must have $m = 1$, i.e. f is birational. Then by the well-known Castelnuovo's genus bound we have $g \le \pi(k^2, 2k-2)$ contrary to the hypothesis on the genus g. □

In the following proposition, we turn to the problem concerning the existence of base-point-free pencil of degree $g-2$ on a double covering of genus two. It should be said that the fact is known and proved in the appendix of [5] with a little bit higher lower-bound on the genus of the given double covering. As we shall see in the proof of the proposition, we use a proof completely different from the one in [5]. And our present proof improves the lower bound on the genus of the given curve a little bit, which could not be detected by the argument in [5]. We also remark the fact that Proposition 2.6 is not a special case of [3, Theorem 0.1]; in [3], the base curve of the covering is a general curve, whereas our base curve in Proposition 2.6 is an arbitrary curve of genus two.

Proposition 2.6. *Let C be a smooth curve of genus $g \geq 11$, which is a double covering of a curve of genus 2. Then there exists a base-point-free pencil of degree g-2 which is not composed with the given double covering.*

Proof. We first recall some of the notations used in [2]. Let $u : C_d \to J(C)$ be the abelian sum map and let θ be the class of the theta divisor in $J(C)$. Let $u^* : H^*(J(C), \mathbb{Q}) \to H^*(C_{g-2}, \mathbb{Q})$ be the homomorphism induced by u. By abusing notation, we use the same letter θ for the class $u^*\theta$. By fixing a point P on C, one has the map $\iota : C_{d-1} \to C_d$ defined by $\iota(D) = D + P$. We denote the class of $\iota(C_{d-1})$ by x.

Let $\pi : C \to E$ be the 2-sheeted covering, genus$(E) = 2$. By the various Martens and Mumford type dimension theorems on the subvarieties of $J(C)$, it is easy to show that $W^1_{g-2}(C)$ is of pure dimension $g - 6 = \rho(g, 1, g-2)$, hence the subvariety C^1_{g-2} of C_{g-2} is of pure dimension $g - 5$. Also it is easy to show that the only components of $W^1_{g-2}(C)$ whose general element has a base point are $\pi^*(W^1_2(E)) + W_{g-6}(C)$ and $\pi^*(W^1_3(E)) + W_{g-8}(C)$ and hence the only components of C^1_{g-2} consisting of divisors whose complete linear series have base points are $\pi^*(E^1_2) + C_{g-6}$ and $\pi^*(E^1_3) + C_{g-8}$ whose class in C^1_{g-2} we denote by γ and η respectively. Because C^1_{g-2} is of pure (and expected) dimension $\rho(g-2, g, 1)+1$, the class c^1_{g-2} of C^1_{g-2} is known (cf. [2, Theorem, p. 326]); $c^1_{g-2} = (\theta^3/6) - (x\theta^2/2)$. Note that γ and η occur with multiplicity 1 in C^1_{g-2}, i.e. C^1_{g-2} is reduced at general points of $\pi^*(E^1_2)+C_{g-6}$ and $\pi^*(E^1_3)+C_{g-8}$; this follows from the description of the tangent space of the scheme C^r_d (cf. [2, Lemma (1.5), p. 162]) and the fact that $h^0(C, K - 2D - \Delta) = 0$ where $D \in \pi^*(E^1_2)$ and $\Delta \in C_{g-6}$ general (or $D \in \pi^*(E^1_3)$ and $\Delta \in C_{g-6}$ general), which can be computed easily.

Let us also recall that given a cycle Z in C_d, the assignments

$$Z \mapsto A_k(Z) := \{E \in C_{d+k} : E - D \geq 0 \text{ for some } D \in Z\},$$

$$Z \mapsto B_k(Z) := \{E \in C_{d-k} : D - E \geq 0 \text{ for some } D \in Z\}$$

induce maps

$$A_k : H^{2m}(C_d, \mathbb{Q}) \to H^{2m}(C_{d+k}, \mathbb{Q}), \qquad B_k : H^{2m}(C_d, \mathbb{Q}) \to H^{2m-2k}(C_{d-k}, \mathbb{Q})$$

and the so-called push–pull formulas for symmetric products hold (cf. [2, p. 367–369]). Thus by the push–pull formulas

$$B_{g-6}(x^{g-5}) = (g-5)x \quad \text{and} \quad B_{g-8}(x^{g-5}) = \frac{(g-5)(g-6)(g-7)}{6}x^3.$$

Denoting $\tilde{\gamma}$ and $\tilde{\eta}$ by the classes of $\pi^*(E^1_2)$ in C_4 and of $\pi^*(E^1_3)$ in C_6, respectively, we will now check that $(\tilde{\gamma} \cdot x)_{C_4} = 1$ and $(\tilde{\eta} \cdot x^3)_{C_6} = 1$, i.e. $\tilde{\gamma}$ and x (resp. $\tilde{\eta}$

and x^3) intersects transversally in C_4 (resp. C_6). Let $D \in \tilde{\gamma} \cap x$ general. Under the natural identification between $T_D(C_4)$ and $H^0(D, \mathcal{O}_D(D))$, the tangent space $T_D(x)$ is the kernel of $H^0(D, \mathcal{O}_D(D)) \to H^0(P, \mathcal{O}_P(D))$ with P the point defining x. One also has $T_D(\tilde{\gamma}) = \{s \in H^0(D, \mathcal{O}_D(D)); Z(s) \in \pi^*(E_2^1)\}$. Since $\tilde{\gamma} = \pi^*(E_2^1) = g_4^1$ is a base-point-free pencil, one finds that $T_D(x) \cap T_D(\tilde{\gamma}) = \{0\}$. For $\tilde{\eta}$ and x^3, define x^3 using P_1, P_2, P_3 with different images on C and fix $D' \in \tilde{\eta} \cap x^3$. Again by noting that the tangent space $T_{D'}(x^3)$ is the kernel of $H^0(D', \mathcal{O}_{D'}(D')) \to H^0(P_1 + P_2 + P_3, \mathcal{O}_{P_1+P_2+P_3}(D'))$ and $T_{D'}(\tilde{\eta}) = \{s \in H^0(D', \mathcal{O}_{D'}(D')); Z(s) \in \pi^*(E_3^1)\}$, one finds that $T_{D'}(x^3) \cap T_{D'}(\tilde{\eta}) = \{0\}$.

Since $(\tilde{\eta} \cdot x^3)_{C_6} = 1$ and $(\tilde{\gamma} \cdot x)_{C_4} = 1$, we have

$$(\gamma \cdot x^{g-5})_{C_{g-2}} = (A_{g-6}(\tilde{\gamma}) \cdot x^{g-5})_{C_{g-2}} = (\tilde{\gamma} \cdot B_{g-6}(x^{g-5}))_{C_4} = (\tilde{\gamma} \cdot (g-5)x)_{C_4} = g - 5$$

and

$$\begin{aligned}(\eta \cdot x^{g-5})_{C_{g-2}} &= (A_{g-8}(\tilde{\eta}) \cdot x^{g-5})_{C_{g-2}} = (\tilde{\eta} \cdot B_{g-8}(x^{g-5}))_{C_6} \\ &= \left(\tilde{\eta} \cdot \frac{(g-5)(g-6)(g-7)}{6} x^3\right)_{C_6} = \frac{(g-5)(g-6)(g-7)}{6} (\tilde{\eta} \cdot x^3)_{C_6} \\ &= \frac{(g-5)(g-6)(g-7)}{6}.\end{aligned}$$

On the other hand $(c_{g-2}^1 \cdot x^{g-5})_{C_{g-2}} = ((\theta^3/6 - x\theta^2/2) \cdot x^{g-5})_{C_{g-2}} = g!/6(g-3)! - g!/2(g-2)!$ by the Poincaré's formula.

Comparing the above intersection numbers we have

$$(\gamma \cdot x^{g-5})_{C_{g-2}} + (\eta \cdot x^{g-5})_{C_{g-2}} < (c_{g-2}^1 \cdot x^{g-5})_{C_{g-2}}$$

and this shows that there exists a component other than $\pi^*(E_2^1) + C_{g-6}$ and $\pi^*(E_3^1) + C_{g-8}$ in C_{g-2}^1 which in turn proves the existence of a divisor of degree $g-2$ which moves in a complete base-point-free pencil and whose complete linear system is not composed with the given involution. □

Acknowledgements

The authors are very grateful to the referee for many valuble comments and suggestions which enabled them to improve the clarity of this paper significantly.

References

[1] R.D.M. Accola, Topics in the theory of Riemann surfaces, Lecture Notes in Mathematics, vol. 1595, Springer, Berlin, 1994.

[2] E. Arbarello, M. Cornalba, P.A. Griffiths, J. Harris, Geometry of Algebraic Curves I, Springer, Berlin, 1985.

[3] E. Ballico, C. Keem, On multiple coverings of irrational curves, Arch. Math. 65 (1995) 151–160.

[4] M. Coppens, Some remarks on the schemes W_d^r, Ann. Mat. Pura Appl. (IV) 157 (1990) 183–197.

[5] M. Coppens, C. Keem, G. Martens, Primitive linear series on curves, Manuscripta Math. 77 (1992) 237–264.
[6] M. Coppens, C. Keem, G. Martens, The primitive length of a general k-gonal curve, Indag. Math., N.S. 5(2) (1994) 145–159.
[7] W. Fulton, R. Lazarsfeld, On the connectedness of degeneracy loci and special divisors, Acta Math. 146 (1981) 271–283.
[8] W. Fulton, J. Harris, R. Lazarsfeld, Excess linear series on an algebraic curve, Proc. Amer. Math. Soc. 92 (1984) 320–322.
[9] A.L. Mayer, Special divisors and the Jacobian variety, Math. Ann. 153 (1964) 163–167.